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Note

On a Conjecture of Fink and Jacobson Concerning k-Domination and k-Dependence

ODILE FAVARON

L.R.I., U.A. 410 C.N.R.S., Bat. 490, Université Paris-Sud, 91405 Orsay, Cedex, France

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A set D of vertices of a graph is k-dependent if every vertex of D is joined to at most k-1 vertices in D. Let $\beta_k(G)$ be the maximum order of a k-dependent set in G. A set D of vertices of G is k-dominating if every vertex not in D is joined to at least k vertices of D. Let $\gamma_k(G)$ be the minimum order of a k-dominating set in G. Here we prove the following conjecture of Fink and Jacobson: for any simple graph G and any positive integer k, $\gamma_k(G) \leqslant \beta_k(G)$.

1. INTRODUCTION

Let $G(V, E)$ be a simple graph of order n and maximum degree Δ. The subgraph induced by a subset A of V is also denoted A. If A is a subset of V, let us denote by $|A|$ the order of A, $m(A)$ the number of its edges, $d_A(x)$ the number of neighbours in A of a vertex x of V, and $\Delta(A) = \sup_{x \in A} d_A(x)$. In [1, 2], Fink and Jacobson gave the following definitions and conjecture:

1.1. DEFINITIONS. A subset D of V is *k-dependent* if $\Delta(D) < k$; $\beta_k(G)$, simply denoted β_k in this paper, is the maximum order of a k-dependent set of G. We notice that the 1-dependent sets are the classical independent sets, that is, $\beta_1 = \beta$; that, if $1 \leqslant k \leqslant j$, then $\beta_k \leqslant \beta_j$; and that $\beta_{\Delta+1} = n$.

A subset D of V is *k-dominating* in G if each vertex of $V - D$ is k-dominated by D, that is if $d_D(x) \geqslant k$ for any x in $V - D$; $\gamma_k(G)$, simply denoted γ_k, is the minimum order of a k-dominating set of G. We notice that the 1-dominating sets are the classical dominating sets, that is $\gamma_1 = \gamma$; that, if $1 \leqslant k \leqslant j$, then $\gamma_k \leqslant \gamma_j$; and that $\gamma_{\Delta+1} = n$.

1.2. *Conjecture* [1, 2]. For any graph G and any positive integer k, $\gamma_k \leqslant \beta_k$.

It is well known that any maximal independent set is a dominating set;

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therefore $\gamma_1 \leqslant \beta_1$. The conjecture was proved for $k=2$ by Fink and Jacobson [1]. We shall prove it for any k.

2. Proof of the Conjecture

We shall prove in Theorem 2.1 the following stronger property: in every graph, and for every $k \geqslant 1$, there exist some subsets which are both k-dependent and k-dominating.

2.1. Theorem. *For any simple graph G and $k \geqslant 1$, every k-dependent set D such that $k|D| - m(D)$ is maximum is a k-dominating set of G.*

Proof. Let D be a k-dependent set such that $k|D| - m(D)$ is maximum. If D is not a k-dominating set of G, let v be a vertex of $V - D$ which is not k-dominated by D; $B = N_D(v)$, the set of the neighbours of v in D (then $0 \leqslant |B| < k$); A the set of the neighbours a of v in D such that $d_D(a) = k-1$; and S a maximal independent set of A. We have $\varnothing \subseteq S \subseteq A \subseteq B \subseteq D$.

The set $C = (D - S) \cup \{v\}$ is still k-dependent. Indeed

$$d_C(v) \leqslant |B| < k.$$

$$d_C(x) \leqslant d_D(x) < k \qquad \text{for any } x \text{ in } D - B.$$

$$d_C(b) \leqslant d_D(b) + 1 < k \qquad \text{for any } b \text{ in } B - A.$$

$$d_C(a) \leqslant d_D(a) = k - 1 \qquad \text{for any } a \text{ in } A - S$$

because every vertex of $A - S$ has at least one neighbour in S (the independent set S being maximal in A).

Furthermore $|C| = |D| - |S| + 1$ and $m(C) = m(D) - (k-1)|S| + |B| - |S| = m(D) - k|S| + |B|$. Thus $k|C| - m(C) = k|D| - m(D) + k - |B| > k|D| - m(D)$, in contradiction with the hypothesis on D. Therefore D is a k-dominating set of G. ∎

2.2. Corollary. *For any simple graph and any positive integer k, $\gamma_k \leqslant \beta_k$.*

Proof. Let D be a k-dependent and k-dominating set of G (such a set exists by the theorem). Then $\gamma_k \leqslant |D| \leqslant \beta_k$. ∎

References

1. J. F. Fink and M. S. Jacobson, On the n-domination number of a graph, to appear.
2. J. F. Fink and M. S. Jacobson, On n-domination, n-dependence, and forbidden subgraphs, to appear.